Exam Geometry 2022

All exercises have equal weight so please try them all. You may refer to theorems and definitions of the lecture notes but are otherwise expected to prove any claims you make. Good luck!

1. (a) Consider the point $q = (1, 1, 1, 1) \in \mathbb{R}^4$ and the U the linear subspace of $\mathbb{R}^4$ spanned by $a = (-1, 1, 1, 1)$ and $b = (1, -1, 1, 1)$. Write down explicit homogeneous coordinates (wrt the standard basis) for a point in $\mathbb{P}^3 \setminus P(q + U)$.
   Solution: We claim that $[-1 : 1 : 1 : 1] = \underline{a}$ is not in $P(p + U) = \{\underline{s} | 0 \neq s \in q+U\}$. Assume to the contrary that for some $t \in \mathbb{R}$ we had $ta \in q+U$ then there would be $x, y \in \mathbb{R}$ such that $ta = q + xa + yb$ and so $q \in U$. However $q \notin U$ because $U \subset \ker \epsilon_1 + \epsilon_2$.

   (b) Imagine a finite dimensional vector space V and prove or give a counter example to the following statement. The intersection of three distinct projective lines in $P(V)$ contains at most one point.
   Solution: The three projective lines are of the form $P(A), P(B), P(C)$ where $A, B, C \subset V$ are two-dimensional linear subspaces of V. Since $P(A \cap B) = P(A) \cap P(B)$ we should consider the case where there are two independent vectors in $A \cap B$. The vectors must be independent because otherwise they would represent the same point in $P(V)$. Since the dimension of A and B is two this means $A = B$ and so the projective lines $P(A)$ and $P(B)$ must coincide. This shows that the intersection of two projective lines cannot contain more than one point. Intersecting with an additional projective line cannot increase the cardinality of the set so the proof is finished.

   (c) Consider the quadratic polynomial $F(x, y, z) = x^2 - 2y^2 - 3z^2$ and the linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ determined by $L(e_1) = e_2$, $L(e_2) = e_3$ and $L(e_3) = e_1$. Find a polynomial G such that the projective transformation $P(L)$ sends $P(X(F))$ to $P(X(G))$.
   Solution: The polynomial is $G = F \circ L^{-1}$ so $G(x, y, z) = F(y, z, x) = y^2 - 2z^2 - 3x^2$. This is because $X(G) = X(F \circ L^{-1}) = L(X(F))$. The last equality is because L is an invertible linear map so $F(x, y, z) = 0$ is equivalent to $(F \circ L^{-1})(L(x, y, z)) = 0$. Finally $P(L)(P(X(F))) = P(L(X(F))) = P(X(G))$ as required.

2. (a) Recall that the geodesic equations for a parametrized curve $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ in the hyperbolic plane take the form $\ddot{\gamma}_1 = 2\gamma_2^{-1}\dot{\gamma}_1\dot{\gamma}_2$ and $\ddot{\gamma}_2 = \gamma_2^{-1}(\dot{\gamma}_2{}^2 - \dot{\gamma}_1{}^2)$. Define two parametrized curves $\beta : (-1, 1) \to \mathbb{H}^2$ and $\gamma : (-1, 1) \to$

$\mathbb{H}^2$ defined by $\gamma(t) = (0, 1+t)$ and $\beta(t) = (1, 1)$. Which of these two parametrized curves β and γ is a hyperbolic geodesic?
Solution: γ is not a hyperbolic geodesic because $\dot{\gamma}(t) = (0, 1)$ so the second geodesic equation is not satisfied: we get $\ddot{\gamma} = 0$ while the right hand side is 1. β is a geodesic because $\dot{\beta} = \ddot{\beta} = 0$ reduces both geodesic equations to $0 = 0$.

(b) $\alpha : (0, 1) \to P$ is a curve defined by $\alpha(t) = (t, \frac{t^2}{2})$ on the Riemannian chart (P, g) where $P = (0, 1) \times (0, 1)$ and the Riemannian metric g is defined by $g_{11}(x, y) = 1$, $g_{22}(x, y) = 2$ and $g_{12}(x, y) = xy$. Find the length of α with respect to this metric.
Solution: $\dot{\alpha}(t) = (1, t)$ and $g(\alpha(t))(\dot{\alpha}(t), \dot{\alpha}(t)) = g(t, \frac{t^2}{2})((1, t), (1, t)) = g_{11}(t, \frac{t^2}{2}) + t^2 g_{22}(t, \frac{t^2}{2}) + 2t g_{12}(t, \frac{t^2}{2}) = 1 + 2t^2 + t^4 = (1 + t^2)^2$. Therefore the length of α is given by the integral $\int_0^1 (1 + t^2) dt = \frac{4}{3}$.

(c) If $W : \mathbb{R}^n \to \mathbb{R}^n$ is an injective linear map then show that W is a Riemannian isometry from the Riemannian chart $(\mathbb{R}^n, g_W)$ to $(\mathbb{R}^n, g_E)$ where g_E is the standard Euclidean metric and g_W is the pull-back metric.
Solution: Since W is linear and injective it is a C^2 bijection with a C^2 inverse and also $dW(p) = W$ for all $p \in \mathbb{R}^n$. To check that W is a Riemannian isometry we need to check that $g_W(p)(a, b) = g_E(W(p))(dW(p)a, dW(p)b)$. By definition of the pull-back metric $g_W(p)(a, b) = \langle W(a), W(b) \rangle = g_E(W(p))(W(a), W(b)) = g_E(W(p))(dW(p)a, dW(p)b)$ using the definition of the Euclidean metric g_E and $dW(p) = W$.

3. (a) Suppose $\varphi \in E(n)$ is a Euclidean isometry and T a k-simplex in $\mathbb{R}^n$. Is it true that $\varphi(T)$ is also a k-simplex? Prove or give a counter example.
Solution: We may assume that $T = [v_0, \dots v_k]$ for some vectors $v_0, \dots v_k \in \mathbb{R}^n$ such that the vectors $u_i = v_i - v_0$, where $i = 1 \dots k$ are linearly independent. Since φ is a linear bijection we find that the vectors $\varphi(u_i)$, $i = 1 \dots k$, are also independent because $\sum_{i=1}^k a_i \varphi(u_i) = 0$ is equivalent to $\sum_{i=1}^k a_i u_i = 0$. We conclude that $\varphi(T) = \varphi(\{\sum_{i=0}^k w_i v_i | w_0 + \cdots + w_k = 1\}) = \{\sum_{i=0}^k w_i \varphi(v_i) | w_0 + \cdots + w_k = 1\} = [\varphi(v_0), \dots, \varphi(v_k)]$

(b) Are the simplicial complexes $K = \{\emptyset, [0]\}$ and $L = \{\emptyset, [0], [e_7]\}$ in $\mathbb{R}^7$ simple homotopy equivalent?
Solution: No because the Euler characteristic χ is preserved under simple homotopy equivalence and $\chi(K) = 1$ while $\chi(L) = 2$.

(c) Is the antipodal map $A : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $A(v) = -v$ an affine rotation?
Solution: An affine rotation is a composition of two reflections whose mirrors intersect. The points on the mirror are fixed by the reflection so points

on the intersection of the two mirrors are fixed by the rotation. In $\mathbb{R}^4$ a mirror for a reflection is a three-dimensional affine subspace. Two three-dimensional affine subspaces cannot intersect in a single point because that would mean their directions only intersect in the origin. It follows that if A would be a rotation then there must be a non-zero vector x in the intersection of the corresponding mirrors and so $A(x) = x$. However $A(x) = -x$ so $x = 0$ gives our contradiction.